Asymptotic Stability of Semilinear Infinite-Dimensional Dissipative Systems

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Outline

1. Introduction
2. Semi-linear Systems
3. Application to Hyperbolic PDEs
4. Conclusion
Consider the system

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + N(x(t)) \\
x(0) &= x_0 \in D(A) \cap F \subset X
\end{aligned}
\]

1. $X$ is a reflexive Banach space.
2. $A$ is a linear operator defined on its domain $D(A)$
3. $N$ is a non-linear operator defined on a closed convex subset $F$. 
Previous Work


1. Nonlinearity defined everywhere
   Tool: m-dissipativity concept

2. Nonlinearity defined on a convex closed subset of $\mathcal{X}$
   Tool: weaker condition of m-dissipativity

\[ \text{conv}(D(A)) \subset \bigcap_{\lambda > 0} \mathcal{R}(I - \lambda A) \]
Previous Work - Basic Result

Let us consider the nonlinear system

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) \\
x(0) &= x_0
\end{aligned}
\]

- \( A \) is dissipative such that
  
  \[
  \text{conv}(D(A)) \subset \cap_{\lambda > 0} \mathcal{R}(I - \lambda A)
  \]

- \((I - \lambda A)^{-1}\) is compact for some \( \lambda > 0 \)
Let us consider the nonlinear system

\[
\begin{cases}
\dot{x}(t) = Ax(t) \\
x(0) = x_0
\end{cases}
\]

- $A$ is dissipative such that

\[
\text{conv}(D(A)) \subset \bigcap_{\lambda > 0} \mathcal{R}(I - \lambda A)
\]

- $(I - \lambda A)^{-1}$ is compact for some $\lambda > 0$

Then for any $x_0 \in \overline{D(A)}$, $x(t, x_0)$ converges to $\omega(x_0)$. 
Let us consider the nonlinear system

\[
\begin{align*}
\dot{x}(t) &= A x(t) \\
x(0) &= x_0
\end{align*}
\]

- $A$ is dissipative such that
  \[
  \text{conv}(D(A)) \subset \cap_{\lambda > 0} \mathcal{R}(I - \lambda A)
  \]
- $(I - \lambda A)^{-1}$ is compact for some $\lambda > 0$

Then for any $x_0 \in \overline{D(A)}$, $x(t, x_0)$ converges to $\omega(x_0)$.

If in addition, $A$ is strictly dissipative, then the system is asymptotically stable.
Consider a semilinear system

\[
\begin{cases}
\dot{x}(t) &= Ax(t) + N(x(t)) \\
x(0) &= x_0 \in D(A) \cap F \subset X
\end{cases}
\]

- \(A\) is closed dissipative and there exists \(\lambda > 0\) such that \((I - \lambda A)^{-1}\) is compact.
- \(N\) is a Lipschitz continuous dissipative operator on a closed convex subset \(F\).
- \(F \subset \mathcal{R}(I - \lambda A)\) for all \(\lambda > 0\).
- \(\lim \inf_{\lambda \to 0^+} \lambda^{-1} d(F, x + \lambda N(x)) = 0\) for \(x \in D(A) \cap F\).
Consider a semilinear system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + N(x(t)) \\
x(0) &= x_0 \in D(A) \cap F \subset X
\end{align*}
\]

- $A$ is closed dissipative and there exists $\lambda > 0$ such that $(I - \lambda A)^{-1}$ is compact.
- $N$ is a Lipschitz continuous dissipative operator on a closed convex subset $F$.
- $F \subset \mathcal{R}(I - \lambda A)$ for all $\lambda > 0$.
- $\liminf_{\lambda \to 0^+} \lambda^{-1} d(F, x + \lambda N(x)) = 0$ for $x \in \overline{D(A) \cap F}$

Then for all $x_0 \in \overline{D(A) \cap F}$, $x(t, x_0) \to \omega(x_0)$. In addition if $N$ is strictly dissipative, the system is asymptotically stable.
Objectives

- Develop stability criteria for
  \[
  \begin{cases}
  \dot{x}(t) &= Ax(t) + N(x(t)) \\
  x(0) &= x_0 \in D(A) \cap F \subset X
  \end{cases}
  \]

- Apply theoretical results to
  \[
  \frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + Mx(t) + f(x(t))
  \]
Result 1

Theorem

Under the assumptions

- $A$ generates a contraction semigroup on $\mathcal{X}$.
- there exists a positive $\lambda$ such that $(I - \lambda A)^{-1}$ is compact.
- $N$ is a Lipschitz continuous dissipative operator on $F$.
- For all $x_0 \in D(A) \cap F$, the system has at least one solution.
Result 1

Theorem

Under the assumptions

- \( A \) generates a contraction semigroup on \( X \).
- there exists a positive \( \lambda \) such that \( (I - \lambda A)^{-1} \) is compact.
- \( N \) is a Lipschitz continuous dissipative operator on \( F \).
- For all \( x_0 \in D(A) \cap F \), the system has at least one solution.

We have

- For any \( x_0 \in D \), \( \lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) \)
- Moreover if \( N \) is strictly dissipative, \( x(t, x_0) \to \bar{x} \)
Sketch of Proof

- \((I - \lambda(A + N))^{-1}\) is compact.

  ▶ \(u_n := (I - \lambda(A + N))^{-1} v_n\)

  ▶ dissipativity of \(A + N \rightarrow u_n\) is bounded.

  ▶ \(w_n := (I - \lambda A) u_n = v_n + \lambda N(u_n)\)
Sketch of Proof

- $(I - \lambda(\mathcal{A} + \mathcal{N}))^{-1}$ is compact.

  ▶ $u_n := (I - \lambda(\mathcal{A} + \mathcal{N}))^{-1}v_n$

  ▶ dissipativity of $\mathcal{A} + \mathcal{N} \to u_n$ is bounded.

  ▶ $w_n := (I - \lambda \mathcal{A})u_n = v_n + \lambda \mathcal{N}(u_n)$

- $\overline{D(\mathcal{A}) \cap F} \subset \cap_{\lambda > 0} \mathcal{R}(I - \lambda \mathcal{A})$

  ▶ Construct a contraction mapping

  ▶ Prove that $(I - \lambda(\mathcal{A} + \mathcal{N}))y = x$ admits a unique solution.
Under the assumptions

- $A$ generates a exponentially stable semigroup on $\mathcal{X}$, i.e. there exist $M \geq 1$ and $\omega < 0$
  \[ \|e^{At}\| \leq Me^{\omega t}, \forall t \geq 0 \]

- there exists a positive $\lambda$ such that $(I - \lambda A)^{-1}$ is compact.

- $N$ is a Lipschitz continuous on $F$, with $l_0 \leq -\omega/M$.

- For all $x_0 \in D(A) \cap F$, the system has at least one solution.
**Result 2**

**Theorem**

*Under the assumptions*

- \( A \) generates a exponentially stable semigroup on \( X \), i.e. there exist \( M \geq 1 \) and \( \omega < 0 \)

\[
\| e^{At} \| \leq Me^{\omega t}, \forall t \geq 0
\]

- there exists a positive \( \lambda \) such that \((I - \lambda A)^{-1}\) is compact.

- \( \mathcal{N} \) is a Lipschitz continuous on \( F \), with \( l_0 \leq -\omega / M \).

- For all \( x_0 \in D(A) \cap F \), the system has at least one solution.

*We have*

- For any \( x_0 \in D \), \( \lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) \)

- Moreover if \( l_0 < -\omega / M \), \( x(t, x_0) \to \bar{X} \)
**Sketch of Proof**

- $M = 1$

\[ \mathcal{A} + \mathcal{N} = \mathcal{A} + l_0 \mathcal{l} + \mathcal{N} - l_0 \mathcal{l} \]
Sketch of Proof

- $M = 1$

\[ A + \mathcal{N} = A + l_0 I + \mathcal{N} - l_0 I \]

- $M \geq 1$

New norm

\[ |x| = \sup \{ \exp(-\omega t) \| e^{At} \|, \ t \geq 0 \} \]

\[ |e^{At}| \leq e^{\omega t} \]
Sketch of Proof

- $M = 1$

\[ \mathcal{A} + \mathcal{N} = \mathcal{A} + l_0 I + \mathcal{N} - l_0 I \]

- $M \geq 1$

New norm

\[ |x| = \sup \{ \exp(-\omega t) \| e^{At} \|, t \geq 0 \} \]

\[ |e^{At}| \leq e^{\omega t} \]
Consider the PDE

$$\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + Mx(t) + f(x(t))$$  \hspace{1cm} (1)

subject to the boundary and initial conditions given by:

$$x(0, t) = C, \text{ and } x(z, 0) = x_0(z)$$  \hspace{1cm} (2)
PDE Model

Consider the PDE

\[
\frac{\partial x}{\partial t} = -\frac{\partial x}{\partial z} + Mx(t) + f(x(t)) \tag{1}
\]

subject to the boundary and initial conditions given by:

\[
x(0, t) = C, \text{ and } x(z, 0) = x_0(z) \tag{2}
\]

- \(x \in H := L^2(0, l)^n\)
- \(f\) is a continuous vector function defined on \(F\)
- \(M = \text{diag}(-\alpha_i), \alpha_i > 0\)
- For all \(x_0 \in D(A_0) \cap F\), (1) has at least one solution.
Properties of the operator

\[ \mathcal{A} = -\frac{d}{dz} + M \cdot I \]

defined on

\[ D(\mathcal{A}) = \{ x \in H : x \text{ is a.c} , \frac{dx}{dz} \in H \text{ and } x(0) = 0 \} \]
Properties of the operator

\[ A = -\frac{d}{dz} + M \cdot I \]

defined on

\[ D(A) = \{ x \in H : x \text{ is a.c, } \frac{dx}{dz} \in H \text{ and } x(0) = 0 \} \]

- \( A \) is m-dissipative.
- \( A \) generates an exponentially stable \( C_0 \)-semigroup.
  Moreover

\[ \| e^{A t} \| \leq e^{-\alpha t}, \text{ with } \alpha = \min\{\alpha_i, i = 1, \ldots n\} \]

- there exists a positive constant \( \lambda \) such that \((I - \lambda A)^{-1}\) is compact.
Results 3&4

Theorem

If the function $f$ is Lipschitz continuous dissipative on $F$, then

\[
\lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) = 0
\]

Moreover if $l_0 < \alpha$, $x(t, x_0) \to x$
Results 3&4

Theorem

*If the function $f$ is Lipschitz continuous dissipative on $F$, then*

- For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0))$
Results 3\&4

**Theorem**

If the function $f$ is Lipschitz continuous dissipative on $F$, then

- For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0))$

- Moreover if $f$ is strictly dissipative, $x(t, x_0) \to \bar{x}$
Theorem

If the function $f$ is Lipschitz continuous dissipative on $F$, then

1. For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0))$
2. Moreover if $f$ is strictly dissipative, $x(t, x_0) \to \bar{x}$

Theorem

If the function $f$ is a Lipschitz continuous on $F$ with a Lipschitz constant $l_0$ such that $l_0 \leq \alpha$, then
Theorem

If the function $f$ is Lipschitz continuous dissipative on $F$, then

- For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0))$

- Moreover if $f$ is strictly dissipative, $x(t, x_0) \to \bar{x}$

Theorem

If the function $f$ is a Lipschitz continuous on $F$ with a Lipschitz constant $l_0$ such that $l_0 \leq \alpha$, then

- For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) = 0$
Results 3&4

**Theorem**

If the function $f$ is Lipschitz continuous dissipative on $F$, then

- For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0))$

Moreover if $f$ is strictly dissipative, $x(t, x_0) \to \bar{x}$

**Theorem**

If the function $f$ is a Lipschitz continuous on $F$ with a Lipschitz constant $l_0$ such that $l_0 \leq \alpha$, then

- For any $x_0 \in D$, $\lim_{t \to \infty} d(x(t, x_0), \omega(x_0)) = 0$

Moreover if $l_0 < \alpha$, $x(t, x_0) \to \bar{x}$
Result 5

Theorem

Assume that there exists $i$ such that $\alpha_i = 0$. If the function $f$ is a Lipschitz continuous on $F$ with a Lipschitz constant $l_0$ such that $l_0 < e^{-1}$, then

$$x(t, x_0) \rightarrow \bar{x}$$

Asymptotic stability
Theorem

Assume that there exists $i$ such that $\alpha_i = 0$. If the function $f$ is a Lipschitz continuous on $F$ with a Lipschitz constant $l_0$ such that $l_0 < e^{-1}$, then

$$x(t, x_0) \to \bar{x}$$

Asymptotic stability

Sketch of Proof For all $\omega \in (\omega_0, 0)$, there exists a constant $M_\omega$ such that

$$\|S_0(t)\| \leq M_\omega e^{\omega t}$$

$$l_0 < \sup_{\omega < 0}(-\omega M_\omega^{-1}) = \sup_{\omega < 0}(-\omega e^{-\omega}) = e^{-1}$$
Summary

- Stability criteria have been developed for two cases
  
  1. Case 1: Lipschitz nonlinearity
  2. Case 2: Dissipative nonlinearity

- Results have been applied to 1st order hyperbolic PDEs