Stabilization of well-posed linear systems by dynamic sampled-data feedback

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Some previous work on **sampled-data control of infinite-dimensional systems**
Some previous work on sampled-data control of infinite-dimensional systems

- Rebarber & Townley, *IEEE TAC*, 1997
- Rebarber & Townley, *SCL*, 1998
- L, Rebarber & Townley, *SICON*, 2005
1 Introduction
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Find necessary and sufficient conditions for the existence of stabilizing linear sampled-data controllers for well-posed infinite-dimensional linear systems
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- $\Sigma$ – well-posed infinite-dimensional continuous-time system
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Objective

Find necessary and sufficient conditions for the existence of stabilizing linear sampled-data controllers for well-posed infinite-dimensional linear systems.

- $\Sigma$ – well-posed infinite-dimensional continuous-time system
- $\Sigma_d$ – discrete-time controller, possibly infinite-dimensional
\[ \tau \text{ – sampling period} \]
\( \tau \) – sampling period

\( \mathcal{H}_\tau \) – zero-order hold
\( \tau - \) sampling period

\( \mathcal{H}_\tau - \) zero-order hold

\( S_\tau - (\text{generalized}) \) sampling operation
\( \tau \) – sampling period

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\( S_\tau \) – (generalized) sampling operation

\( v, v_d \) – closed-loop inputs
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\( v, v_d \) – closed-loop inputs

\( y, y_d \) – closed-loop outputs
2 Formulation of the problem
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Continuous-time system \( \Sigma \)
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Continuous-time system $\Sigma$

- $X$ – state space of well-posed system $\Sigma$
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- MIMO – $m$ inputs, $p$ outputs
Continuous-time system $\Sigma$

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- MIMO – $m$ inputs, $p$ outputs
- $A$, $B$, $C$ – generating operators of $\Sigma$
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Continuous-time system $\Sigma$

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- $T = (T_t)_{t \geq 0}$ – $C_0$-semigroup generated by $A$
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- $T = (T_t)_{t \geq 0}$ – $C_0$-semigroup generated by $A$
- $G$ – transfer function of $\Sigma$

\[
\begin{align*}
\dot{x} &= Ax + Bu, \quad x(0) = x^0 \in X, \\
y &= C\lambda (x - (\lambda I - A)^{-1} Bu) + G(\lambda)u,
\end{align*}
\]

where

$\text{Re } \lambda > \text{exponential growth constant of } T$
Discrete-time system $\Sigma_d$
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- $X_d$ – state space of discrete-time system $\Sigma_d$
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Discrete-time system $\Sigma_d$

- $X_d$ – state space of discrete-time system $\Sigma_d$
- MIMO – $p$ inputs, $m$ outputs
- $A_d$, $B_d$, $C_d$ – generating bounded operators of $\Sigma_d$

\[\begin{align*}
  x_d(k+1) &= A_d x_d(k) + B_d u_d(k), \quad x_d(0) = x_d^0 \in X_d, \\
  y_d(k) &= C_d x_d(k) \end{align*}\]  

$\{ \Sigma_d \}$
Hold and sampling operations
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- **Zero-order hold** operator $\mathcal{H}_\tau$ maps discrete-time signals to continuous-time signals

\[
(\mathcal{H}_\tau f_d)(t) = f_d(k) \quad \forall \ t \in [k\tau, (k-1)\tau)
\]
Hold and sampling operations

- **Zero-order hold** operator $\mathcal{H}_\tau$ maps discrete-time signals to continuous-time signals

  \[(\mathcal{H}_\tau f_d)(t) = f_d(k) \quad \forall t \in [k\tau, (k-1)\tau)\]

- **Generalized sampling** operator $S_\tau$ maps continuous-time signals to discrete-time signals

  \[(S_\tau f)(k) = \int_0^\tau w(t) f(k\tau + t)dt \quad \forall f \in L^2_{\text{loc}}(\mathbb{R}^+)\]

  where $w \in L^2(0, \tau)$. 

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Why generalized sampling?
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**Feedback interconnection** of $\Sigma$ and $\Sigma_d$
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\begin{align*}
  u &= v - \mathcal{H}_\tau y_d \\
  u_d &= v_d + S_\tau y
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\begin{align*}
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    u_d &= v_d + S_\tau y
\end{align*}
\]

For the resulting sampled-data system the abbreviation SDS will be used.
3 Main result
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Stability concept
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Stability concept

SDS is exponentially $L^q/l^q$-input-to-state stable if there exist $\Gamma > 0$ and $\gamma > 0$ such that
3 Main result

Stability concept

SDS is **exponentially $L^q/l^q$-input-to-state stable** if there exist $\Gamma > 0$ and $\gamma > 0$ such that

$$
\left\| \begin{pmatrix} x(k\tau + t) \\ x_d(k) \end{pmatrix} \right\| \leq \Gamma \left[ e^{-\gamma(k\tau+t)} \left\| \begin{pmatrix} x^0 \\ x^0_d \end{pmatrix} \right\| + \| v \|_{L^q} + \| v_d \|_{l^q} \right]
$$

for all $k \in \mathbb{Z}_+, t \in [0, \tau), v \in L^q(\mathbb{R}_+)$ and $v_d \in l^q(\mathbb{Z}_+)$. 
Assumptions
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**(A1)** There exists $\alpha > 0$ such that the spectrum of $A$ in $\mathbb{C}_{-\alpha}$ consists of finitely many isolated eigenvalues of $A$ with finite algebraic multiplicities.
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If (A1) holds, then there exists a projection $\Pi : X \to X$ such that

- $X^+ := \Pi X$ and $X^- := (I - \Pi)X$ are $T$-invariant.
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Here

$$\Pi := \frac{1}{2\pi i} \int_{\phi} (sI - A)^{-1} ds,$$
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Here

$$\Pi := \frac{1}{2\pi i} \int_{\Phi} (sI - A)^{-1} ds,$$

where $\Phi$ is a simple closed curve enclosing $\sigma(A) \cap \overline{\mathbb{C}}_0$.
Assumptions continued
Assumptions continued

Define

\[ A^+ := A|_{X^+}, \quad B^+ := \nabla B, \quad C^+ := C|_{X^+}, \]
\[ T^+_t := T_t|_{X^+}, \quad T^-_t := T_t|_{X^-} \]
Assumptions continued

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Note that

\[ \sigma(A^+) = \sigma(A) \cap \overline{C}_0, \quad T^+_t = e^{A^+t} \]
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(A2) \((T^-_t)_{t \geq 0}\) is exponentially stable
Assumptions continued

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(A2) \((T^-_t)_{t \geq 0}\) is exponentially stable

(A3) \((T^+_\tau, B^+)\) is controllable & \((C^+, T^+_\tau)\) is observable
Assumptions continued

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(A3) \((T^+_\tau, B^+))\) is controllable & \((C^+, T^+_\tau))\) is observable

(A4) \(2k\pi i/\tau \notin \sigma(A^+)\) for all \(k \in \mathbb{Z}, k \neq 0\)
Assumptions continued

Define

\[ A^+ := A|_{X^+}, \quad B^+ := B, \quad C^+ := C|_{X^+}, \]
\[ T^+_t := T_t|_{X^+}, \quad T^-_t := T_t|_{X^-} \]

Note that

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(A5) \(\int_0^\tau w(t)e^{\lambda t}dt \neq 0\) for every \(\lambda \in \sigma(A^+)\)
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Let $2 \leq q \leq \infty$.

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(a) (A1)-(A5) hold.

(b) There exists a discrete-time controller $(A_d, B_d, C_d)$ such that SDS is exponentially $L^q/l^q$-input-to-state stable.
Theorem

Let \( 2 \leq q \leq \infty \).

The following statements are equivalent.

(a) \((A1)-(A5)\) hold.

(b) There exists a discrete-time controller \((A_d, B_d, C_d)\) such that SDS is exponentially \(L^q/l^q\)-input-to-state stable.

(c) There exists a finite-dimensional discrete-time controller \((A_d, B_d, C_d)\) such that SDS is exponentially \(L^q/l^q\)-input-to-state stable.
4 Discussion of main result
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Recall assumptions (A3)-(A5):
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\textbf{(A3)} \quad (T^+_\tau, B^+) is controllable & (C^+, T^+_\tau) is observable

\textbf{(A4)} \quad 2k\pi i/\tau \notin \sigma(A^+) for all \( k \in \mathbb{Z}, k \neq 0 \)

\textbf{(A5)} \quad \int_0^\tau w(t)e^{\lambda t}dt \neq 0 \text{ for every } \lambda \in \sigma(A^+)
4 Discussion of main result

Recall assumptions (A3)-(A5):

(A3) \((T_+^+, B^+)\) is controllable & \((C^+, T_+^+)\) is observable

(A4) \(2k\pi i/\tau \notin \sigma(A^+)\) for all \(k \in \mathbb{Z}, k \neq 0\)

(A5) \(\int_0^\tau w(t)e^{\lambda t}dt \neq 0\) for every \(\lambda \in \sigma(A^+)\)

Special case: if \(w(t) \equiv \text{const}\), then (A4) \(\Rightarrow\) (A5)
4 Discussion of main result

Recall assumptions (A3)-(A5):

(A3) $(T^+_{\tau}, B^+)$ is controllable & $(C^+, T^+_{\tau})$ is observable

(A4) $2k\pi i/\tau \not\in \sigma(A^+)$ for all $k \in \mathbb{Z}$, $k \neq 0$

(A5) $\int_0^\tau w(t)e^{\lambda t} dt \neq 0$ for every $\lambda \in \sigma(A^+)$

Special case: if $w(t) \equiv \text{const}$, then (A4) $\Rightarrow$ (A5)

Alternative assumptions for (A3) and (A4):
Recall assumptions (A3)-(A5):

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Special case: if \(w(t) \equiv \text{const}\), then (A4) \(\Rightarrow\) (A5)

Alternative assumptions for (A3) and (A4):

(A3') \((A^+, B^+)\) is controllable & \((C^+, A^+)\) is observable
4 Discussion of main result

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(A3) \((T^+_\tau, B^+)\) is controllable & \((C^+, T^+_\tau)\) is observable

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Alternative assumptions for (A3) and (A4):

(A3') \((A^+, B^+)\) is controllable & \((C^+, A^+)\) is observable

(A4') \(\tau(\lambda - \mu) \neq 2k\pi i\) for all \(k \in \mathbb{Z}, k \neq 0\), and all \(\lambda, \mu \in \sigma(A^+)\)
By standard results from finite-dimensional sampled-data theory:
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\[(A3') \& (A4') \Rightarrow (A3)\]
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- In **SISO** case: 
  \((A3') \& (A4') \Leftrightarrow (A3)\)
By standard results from finite-dimensional sampled-data theory:

- \((A3') \& (A4') \Rightarrow (A3)\)

- In **SISO** case: \((A3') \& (A4') \Leftrightarrow (A3)\)

- \((A4') \Rightarrow (A4), \ (A4') \not\equiv (A4)\)
By standard results from finite-dimensional sampled-data theory:

- \((A3') \& (A4') \Rightarrow (A3)\)

- In SISO case: \((A3') \& (A4') \Leftrightarrow (A3)\)

- \((A4') \Rightarrow (A4), \ (A4') \not\Leftrightarrow (A4)\)

- In particular, in SISO case: \((A3) \Rightarrow (A4)\)
5 Proof of main result: some comments
Sample-hold discretization of $\Sigma$
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Sample-hold discretization of Σ

Define **bounded** operators $B_τ$, $C_τ$ and $D_τ$
5 Proof of main result: some comments

Sample-hold discretization of $\Sigma$

Define bounded operators $B_\tau$, $C_\tau$ and $D_\tau$

- $B_\tau \xi := \int_0^\tau T_t B \xi dt \quad \forall \xi \in \mathbb{R}^m$
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- $B_\tau\xi := \int_0^\tau T_t B \xi dt \quad \forall \xi \in \mathbb{R}^m$

- $C_\tau\zeta := \int_0^\tau w(t) C_{\chi} T_t \zeta dt \quad \forall \zeta \in X$
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- $C_\tau \zeta := \int_0^\tau w(t) C \Lambda T_t \zeta dt \quad \forall \zeta \in X$
- $D_\tau \xi := \int_0^\tau w(t)(G \xi)(t) dt \quad \forall \xi \in \mathbb{R}^m$,
5 Proof of main result: some comments

Sample-hold discretization of $\Sigma$

Define bounded operators $B_\tau$, $C_\tau$ and $D_\tau$

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- $D_\tau \xi := \int_0^\tau w(t)(G \xi)(t) dt \quad \forall \xi \in \mathbb{R}^m,$

where $G$ is the i/o-operator of $\Sigma$. 
Lemma

Let the input $u$ of $\Sigma$ be of the form $u = H_\tau f_d$, where $f_d$ is a discrete-time signal. Let $x$ and $y$ be the corresponding state and output functions of $\Sigma$, respectively.
Lemma

Let the input $u$ of $\Sigma$ be of the form $u = \mathcal{H}_\tau f_d$, where $f_d$ is a discrete-time signal. Let $x$ and $y$ be the corresponding state and output functions of $\Sigma$, respectively.

Then, for all $k \in \mathbb{Z}_+$,

\[
\begin{align*}
x((k + 1)\tau) &= T_\tau x(k\tau) + B_\tau f_d(k) \\
(S_\tau y)(k) &= C_\tau x(k\tau) + D_\tau f_d(k)
\end{align*}
\]
Lemma

Let the input \( u \) of \( \Sigma \) be of the form \( u = H_\tau f_d \), where \( f_d \) is a discrete-time signal. Let \( x \) and \( y \) be the corresponding state and output functions of \( \Sigma \), respectively.

Then, for all \( k \in \mathbb{Z}_+ \),

\[
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x((k + 1)\tau) &= T_\tau x(k\tau) + B_\tau f_d(k) \\
(S_\tau y)(k) &= C_\tau x(k\tau) + D_\tau f_d(k)
\end{align*}
\]

The discrete-time system \((T_\tau, B_\tau, C_\tau, D_\tau)\) is the so-called sample-hold discretization of \( \Sigma \).
Proposition
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Let $2 \leq q \leq \infty$.

SDS is exponentially $L^q/l^q$-input-to-state stable if and only if

$\Sigma_d = (A_d, B_d, C_d)$ stabilizes $(T_\tau, B_\tau, C_\tau, D_\tau)$ in the sense that the operator
Proposition

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$
\Sigma_d = (A_d, B_d, C_d)$ stabilizes $(T_\tau, B_\tau, C_\tau, D_\tau)$ in the sense that the operator

$$
\Delta := \begin{pmatrix}
T_\tau & -B_\tau C_d \\
B_d C_\tau & A_d - B_d D_\tau C_d
\end{pmatrix}
$$

is power-stable (that is, the spectral radius of $\Delta$ is smaller than 1).
Sketch proof of (a) \implies (c)
Sketch proof of \((a) \Rightarrow (c)\)

Transfer function of sample/hold discretization:

\[
G_\tau(\bar{z}) = C_\tau(\bar{z}I - T_\tau)^{-1}B_\tau + D_\tau
\]
Sketch proof of (a) ⇒ (c)

Transfer function of sample/hold discretization:

\[ G_\tau(z) = C_\tau (zI - T_\tau)^{-1} B_\tau + D_\tau \]

Decomposition:

\[ G_\tau = G^-_\tau + G^+_\tau \]
Sketch proof of (a) $\Rightarrow$ (c)

Transfer function of sample/hold discretization:

$$G_T(z) = C_T(zI - T_T)^{-1}B_T + D_T$$

Decomposition: $G_T = G_T^- + G_T^+$

Here

$$G_T^-(z) = C_T^-(zI - T_T^-)^{-1}B_T^- + D_T,$$
$$G_T^+(z) = C_T^+(zI - T_T^+)^{-1}B_T^+,$$

where $B_T^+$, $B_T^-$ etc are defined in the usual way
By (A1) and (A2)
By (A1) and (A2)

- $G_\tau^+$ rational and strictly proper
By (A1) and (A2)

- $G_{\tau}^+$ rational and strictly proper

- $G_{\tau}^- \in H_\rho^\infty$ for some $\rho \in (0, 1)$
By (A1) and (A2)

- $G^+_\tau$ rational and strictly proper
- $G^-_{\tau} \in H^\infty_\rho$ for some $\rho \in (0, 1)$

where

$$H^\infty_\rho := \{\text{functions holomorphic & bounded on } |z| > \rho\}$$
By (A1) and (A2)

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- $G^-_{\tau} \in H^{\infty}_\rho$ for some $\rho \in (0, 1)$

where

$$H^{\infty}_\rho := \{\text{functions holomorphic \& bounded on } |z| > \rho\}$$

Well-known frequency-domain result guarantees existence of a strictly proper rational $G_d$ such that

$$F(G_{\tau}, G_d) := \begin{pmatrix}
(I + G_{\tau}G_d)^{-1} & G_{\tau}(I + G_dG_{\tau})^{-1} \\
G_d(I + G_{\tau}G_d)^{-1} & (I + G_dG_{\tau})^{-1}
\end{pmatrix} \in H^{\infty}$$
Let \((A_d, B_d, C_d)\) be a stabilizable and detectable realization of \(G_d\).
Let \((A_d, B_d, C_d)\) be a stabilizable and detectable realization of \(G_d\).

\[ F(G_t, G_d) \in H^\infty \] means that \((A_d, B_d, C_d)\) stabilizes \((A_t, B_t, C_t, D_t)\) in the sense of \(l^2\)-stability.
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- \(F(G_\tau, G_d) \in H^\infty\) means that \((A_d, B_d, C_d)\) stabilizes \((A_\tau, B_\tau, C_\tau, D_\tau)\) in the sense of \(l^2\)-stability.

By Proposition, it is sufficient to show that

\[
\Delta = \begin{pmatrix}
T_\tau & -B_\tau C_d \\
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\end{pmatrix}
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is power stable.
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\(F(G_\tau, G_d) \in H^\infty\) means that \((A_d, B_d, C_d)\) stabilizes \((A_\tau, B_\tau, C_\tau, D_\tau)\) in the sense of \(l^2\)-stability.

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\]

is power stable.

Power-stability of \(\Delta\) will follow from \(l^2\)-stability, provided that \((T_\tau, B_\tau)\) is stabilizable and \((C_\tau, T_\tau)\) is detectable.
Let \((A_d, B_d, C_d)\) be a stabilizable and detectable realization of \(G_d\).

\[ F(G_\tau, G_d) \in H^\infty \] means that \((A_d, B_d, C_d)\) stabilizes \((A_\tau, B_\tau, C_\tau, D_\tau)\) in the sense of \(l^2\)-stability.

By Proposition, it is sufficient to show that

\[
\Delta = \begin{pmatrix}
T_\tau & -B_\tau C_d \\
B_d C_\tau & A_d - B_d D_\tau C_d
\end{pmatrix}
\]

is power stable.

Power-stability of \(\Delta\) will follow from \(l^2\)-stability, provided that \((T_\tau, B_\tau)\) is stabilizable and \((C_\tau, T_\tau)\) is detectable.

The latter is a consequence of (A3)-(A5).
Comment on (b) ⇒ (a)
Comment on \((b) \Rightarrow (a)\)

It follows from a general result by Rebarber & Townley (\textit{IEEE TAC}, 1997) that (A1) and (A2) are necessary for (b).
Comment on \((b) \Rightarrow (a)\)

It follows from a general result by Rebarber & Townley (\textit{IEEE TAC}, 1997) that \((A1)\) and \((A2)\) are necessary for \((b)\).

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