Optimal Actuator Location

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Examples of Control Systems Modelled by PDE's

Beam

Plate (WPI)

Acoustic Noise in Duct
State-space Description

\[ \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0 \]
\[ y(t) = Cz(t). \]

- \( A \) generates a strongly continuous semigroup \( S(t) \) on \( \mathcal{H} \)
- \( z(t) \) is state, state-space is \( \mathcal{H} \)
- \( B \in \mathcal{L}(\mathcal{U}, \mathcal{H}), \ C \in \mathcal{L}(\mathcal{H}, \mathcal{Y}) \)
Linear Quadratic (LQ) Control

\[
\inf_{u \in L_2(0, \infty; U)} \int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), u(t) \rangle \, dt
\]

Assumption: \((A, B)\) is stabilizable
There exists \(K\) so that semigroup generated by \(A - BK\) is exponentially stable.

Assumption: \((A, C)\) is detectable
There exists \(F\) so that semigroup generated by \(A - FC\) is exponentially stable.
Linear Quadratic (LQ) Control

\[
\inf_{u \in L_2(0, \infty; \mathcal{U})} \int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), u(t) \rangle \, dt \equiv J(u, z_0)
\]

**Theorem**

If \((A, B)\) is stabilizable and \((A, C)\) is detectable then there exists a unique \(\Pi \geq 0\) such that for all \(z \in D(A)\),

\[
(\Pi A + A^* \Pi + C^* C - \Pi BB^* \Pi) z = 0,
\]

- \(\inf_{u \in L_2(0, \infty; \mathcal{U})} J(u, z_0) = \langle z_0, \Pi z_0 \rangle\)
- Optimal control is \(u(t) = -B^* \Pi z(t)
\]
- \(A - BK\) generates an exponentially stable semigroup.
Calculation of Linear Quadratic Regulator

Operator ARE

\[ A^* \Pi + \Pi A - \Pi BB^* \Pi + C^* C = 0 \]

- Need to approximate solution
- Approximate \( A, B, C \) by \( A_n, B_n, C_n \)
- Let \( S_n(t) \) indicate the semigroup generated by \( A_n \).
- Approximation \( \Pi_n \) and hence \( K_n \) used to control original system
Convergence of $\Pi_n$

Assume that for each $z \in \mathcal{H}$ and $y \in cY$,

(A1i) $\|S_n(t)P_n z - S(t)z\| \to 0$

(A1ii) $\|S^*_n(t)P_n z - S^*(t)z\| \to 0$ (both uniformly in $t$ on bounded intervals)

(A2i) $\|B_n u - B u\| \to 0$, $\|C_n P_n z - C z\| \to 0$

(A2ii) $\|B^*_n z - B^* z\| \to 0$, $\|C^*_n y - C^* y\| \to 0$

(A3i) $(A_n, B_n)$ is uniformly exponentially stabilizable

(A3ii) $(A_n, C_n)$ is uniformly exponentially detectable.

Then for all $z \in \mathcal{H}$,

- $\|\Pi_n P_n z - \Pi z\| \to 0$

- there exists constants $M_2 \geq 1$, $\alpha_2 > 0$, independent of $n$, such that

$$\|e^{(A_n - B_n K_n)t}\| \leq M_2 e^{-\alpha_2 t}.$$
Convergence of Riccati operator

Cost arbitrarily close to optimal can be achieved:

- For sufficiently large $n$, semigroups generated by $A - BK_n$ are uniformly exponentially stable
- Cost with feedback $K_n$ converges to optimal:

$$J(-K_n z(t), z_0) \to \langle \Pi z_0, z_0 \rangle.$$
Formulation of LQ-Optimal Actuator Location Problem

- Consider one actuator with location in some closed and bounded set $\Omega \subset \mathbb{R}^q$.
- Indicate the corresponding input operator by $B(r)$.
- For each $r$ have an optimal control problem $J^r(u, z_0)$.
- Choose actuator location $r$ to minimize the response to the worst initial condition:

$$
\max_{z_0 \in \mathcal{H}} \min_{\|z_0\|=1} J^r(u, z_0) = \max_{z_0 \in \mathcal{H}} \langle \Pi(r)z_0, z_0 \rangle
\min_{\|z_0\|=1} \langle \Pi(r)z_0, z_0 \rangle = \|\Pi(r)\|.
$$
Well-posedness of optimal actuator location

Theorem

If

- For any $r_0$, $\lim_{r \to r_0} \|B(r) - B(r_0)\|$,
- $(A, B(r))$ are all stabilizable, $(A, C)$ is detectable
- $B$ is a compact operator,

then

$$\lim_{r \to r_0} \|\Pi(r) - \Pi(r_0)\| = 0.$$ 

Also, there exists $\hat{r}$ such that

$$\|\Pi(\hat{r})\| = \inf_{r \in \Omega^m} \|\Pi(r)\| = \hat{\mu}.$$
Optimal Actuator Location for Simply Supported Beam

PDE

\[
\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = b(x)u(t), \quad t \geq 0, 0 < x < 1,
\]

\[
w(0, t) = 0, \quad w_{xx}(0, t) = 0, \quad w(1, t) = 0, \quad w_{xx}(1, t) = 0.
\]

\[
b(r) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2} 
\end{cases}
\]

- reduce state uniformly: \( C = I \)
- eigenfunction approximations
Optimal Actuator Location for Simply Supported Beam

\[
\begin{aligned}
\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} &= b(x)u(t), \quad t \geq 0, 0 < x < 1, \\
w(0, t) &= 0, \ w_{xx}(0, t) = 0, \ w(1, t) = 0, \ w_{xx}(1, t) = 0.
\end{aligned}
\]

\[
b(r) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2}
\end{cases}
\]

- reduce state uniformly: \( C = I \)
- eigenfunction approximations
∥Πₙ∥, simply supported beam, $C = I$, actuator at $x = 0.5$
Optimal performance ($\|\Pi_n\|$, $C = I$)
Optimal actuator location ($\|\Pi_n\|$, $C = I$)
Example

On the Hilbert space $\mathcal{R} \times L_2$, 

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -I \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2}I \end{bmatrix} \]

The solution to the ARE

\[ A^* \Pi + \Pi A - \Pi BB^* \Pi + C^* C = 0 \]

is

\[ \Pi = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}. \]

Solution is not compact.
Example

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The solution to the ARE 

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + C^* C = 0$$

is 

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & I \end{bmatrix}.$$ 

Solution is not compact.
Compact solution to ARE

**Theorem**

If \((A, B)\) is stabilizable, \((A, C)\) is detectable, \(B, C\) are compact operators, then \(\Pi\) is a compact operator.

**Theorem**

If

- \((A, B)\) is stabilizable and \((A, C)\) is detectable
- \((A_n, B_n, C_n)\) satisfies (A1)-(A3) for each \(r\)
- \(B\) and \(C\) are compact operators

then

\[
\lim_{n \to \infty} \|\Pi_n P_n - \Pi\| = 0.
\]
Compact solution to ARE

**Theorem**

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then

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\lim_{n \to \infty} \| \Pi_n P_n - \Pi \| = 0.
\]
**Theorem**

**Assume**

- \((A, B(r))\) are all stabilizable, \((A, C)\) is detectable,
- \((A_n, B_n, C_n)\) satisfies (A1)-(A3) for each \(r\),
- \(B, C\) are compact operators,
- for any \(r_0\), \(\lim_{r \to r_0} \| B(r) - B(r_0) \|\).

**Then**

\[
\hat{\mu} = \lim_{n \to \infty} \hat{\mu}_n,
\]

and there exists a subsequence \(\{\hat{r}_m\}\) of \(\{\hat{r}_n\}\) such that

\[
\hat{\mu} = \lim_{m \to \infty} \| \Pi(\hat{r}_m) \|.
\]
Optimal performance $\|\Pi_n\|$ for simply supported beam, state weight is $C = [I \ 0]$
Optimal actuator location, state weight is $C = [I \ 0]$
Analytic Semigroups

Definition

A semigroup $S(t)$ is analytic if $t \rightarrow S(t)$ is analytic in some sector $|\text{arg} t| < \theta$.

- Eigenvalues of $A$ lie in a sector $|\text{arg} \lambda| < \pi - \epsilon$
- Additional smoothness in the solution
- Heat equation and most other parabolic partial differential equations lead to an analytic semigroup
- Weakly damped wave and beam equations not associated with analytic semigroups
Solution to ARE for analytic semigroups

Theorem

Consider a smooth bounded domain \( \Omega \subset \mathbb{R}^N \). Let \( A \) be an elliptic operator on \( L_2(\Omega) \) of order \( 2m \), subject to appropriate boundary conditions. Then \( A \) generates an analytic semigroup on \( L_2(\Omega) \) and the solution \( \Pi \) to the ARE is a compact operator.

- Uniform convergence of \( \Pi_n \) to \( \Pi \) occurs for typical approximation schemes
- Applies to unbounded control operators \( B \) and observation \( C \)
- Convergence of optimal actuator locations, even for non-compact weight on state
Example: Diffusion

PDE

\[
\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + b(x)u(t) \quad 0 < x < 1,
\]

\[
b(x) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2} 
\end{cases}
\]

\[
\frac{\partial z}{\partial x}(0, t) = 0, \quad \frac{\partial z}{\partial x}(1, t) = 0.
\]

Determine best location of actuator to minimize

\[
J^r(u, z_0) = \int_0^\infty 1000 \int_0^1 z(x, t)^2 dx + u(t)^2 dt
\]

with respect to worst possible initial condition.
Diffusion example, Optimal Actuator Location, $C=I$
Performance vs Location

\[ \|P_N(r)\| \]

\[ x \]

Graph showing the relation between \( \|P_N(r)\| \) and \( x \) for varying values of \( r \). The graph plots \( \|P_N(r)\| \) against \( x \) from 0 to 1 with markers indicating the function's behavior across this range.
Another Measure of Performance

- If $z(0)$ is random, with zero mean and variance $V$ then expected cost is
  
  \[ \text{trace} \left( V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \right), \]

  or since $\Pi$ is self-adjoint and non-negative,

  \[ \| \left( V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \right) \|_1 \]

  where $\| \cdot \|_1$ indicates the nuclear norm.

- Let $V = I$ to simplify.

Performance

Performance for location $r$ is $\mu(r) = \| \Pi(r) \|_1$ and the optimal performance

\[ \hat{\mu} = \inf_{r \in \Omega} \| \Pi(r) \|_1. \]

(If $U$ and $Y$ and both finite-dimensional, then $\Pi(r)$ is nuclear.)
Another Measure of Performance

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Performance

Performance for location \( r \) is \( \mu(r) = \| \Pi(r) \|_1 \) and the optimal performance

\[
\hat{\mu} = \inf_{r \in \Omega^m} \| \Pi(r) \|_1.
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(If \( \mathcal{U} \) and \( \mathcal{Y} \) and both finite-dimensional, then \( \Pi(r) \) is nuclear.)
Another Measure of Performance

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Performance

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$$\hat{\mu} = \inf_{r \in \Omega^m} \| \Pi(r) \|_1.$$
Well-posedness of optimal actuator location

**Theorem**

Assume that \((A, B(r))\) are all stabilizable and that \((A, C)\) is detectable where \(C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})\) and \(\mathcal{U}\) and \(\mathcal{Y}\) are finite-dimensional. If for any \(r_0 \in \Omega_m\),

\[
\lim_{r \to r_0} \|B(r) - B(r_0)\| = 0,
\]

then the corresponding Riccati operators \(\Pi(r)\) are continuous functions of \(r\) in the nuclear norm:

\[
\lim_{r \to r_0} \|\Pi(r) - \Pi(r_0)\|_1 = 0
\]

and there exists an optimal actuator location \(\hat{r}\) such that

\[
\|\Pi(\hat{r})\|_1 = \inf_{r \in \Omega_m} \|\Pi(r)\|_1 = \hat{\mu}.
\]
Theorem

Assume that \((A, B)\) is stabilizable and \((A, C)\) is detectable, and that \(U\) and \(Y\) are finite-dimensional. Let \((A_n, B_n, C_n)\) be a sequence of approximations to \((A, B, C)\) such that assumptions (A1)-(A3) are satisfied. Then

\[
\lim_{n \to \infty} \| \Pi_n P_n - \Pi \|_1 = 0.
\]
**Theorem**

Assume a stabilizable/detectable family of control systems $(A, B(r), C)$ with *finite-dimensional* input space $\mathcal{U}$ and output space $\mathcal{Y}$ such that for any $r_0 \in \Omega$, \[
\lim_{r \to r_0} \| B(r) - B(r_0) \| = 0.
\]

*If the approximation scheme satisfies (A1)-(A3) then* \[
\hat{\mu} = \lim_{n \to \infty} \hat{\mu}_n,
\]

*and there exists a subsequence $\{\hat{r}_m\}$ of $\{\hat{r}_n\}$ such that* \[
\hat{\mu} = \lim_{m \to \infty} \| \Pi(\hat{r}_m) \|_1.
\]
$\|\Pi_n\|_1$ vs $n$ with actuator at $x = 0.5$, state weight $C = I$
Optimal performance ($\|\Pi_n\|_1$), $Cz = w(0.5)$
Optimal actuator locations, $Cz = w(.5)$
approximated controller must control original PDE system
need to consider infinite-dimensional controller in constructing approximation
extensive body of results for LQ control for theory and approximation
may not have uniform convergence of ARE solution if $B$ and $C$ not compact
uniform convergence of Riccati operators important for optimal actuator location
compactness of $B$ and $C$, along with standard approximation assumptions, imply uniform convergence of ARE solution
if input and output spaces finite-dimensional, obtain trace-norm convergence
optimal sensor location problem is dual