

# Local Cuts and Two-Period Convex Hull Closures for Big Bucket lot-sizing Problems <sup>\*</sup>

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## 1 Introduction

Despite the significant attention that they have drawn over the years, big bucket lot-sizing problems remain notoriously difficult to solve. The authors have previously presented evidence that what make these problems difficult are the embedded single-machine, single-level, multi-period submodels [1]. We therefore consider the simplest such submodel, a multi-item, two-period capacitated model.

Recently, there has been a large body of MIP research that has generated promising results on the “closures” of general cutting planes and some particular polyhedra. Even partially achieving some elementary closures has helped researchers to be able to close duality gaps efficiently and solve some problems that were never solved before (e.g., [5, 3]). “Closure” in this perspective can be defined as the polyhedron defined by all the valid inequalities of a given type. In this research we propose a methodology that can approximate the intersection of the convex hulls of all possible two period subproblems; we therefore approximate the “closure” of all valid inequalities that can be generated in this way.

## 2 Separating over convex hulls of two-period submodels

The original problems we seek to solve are general multi-level, multi-item lot-sizing problems. For space reasons, we do not give the complete formulation of a general form of these problems; rather, we present here the formulation of the feasible region of the two-period submodel. We note that the original problems that we seek to solve are much more complicated than this, and contain many of the instances of the above model as a substructure. This submodel defines a single-machine problem, and considers only a single stock variable  $s^i$  for each item. The parameter  $\tilde{d}$  represents the remaining cumulative demand, e.g.,  $\tilde{d}_1^i$  is the demand for  $i$  in periods 1 and 2. The parameter  $ST_t^i$  is

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the setup time for item  $i$  in period  $t$ , and  $C_t$  is the total production capacity in period  $t$ . Note also that the following formulation, which we refer to as  $X^{2PL}$ , can be seen as a multi-item extension of the bottleneck flow formulation studied by [2] when  $NT = 2$ .

$$x_{t'}^i \leq (C_{t'} - ST_{t'}^i)y_{t'}^i \quad i = [1, \dots, NI], t' = 1, 2 \quad (1)$$

$$x_{t'}^i \leq \tilde{d}_{t'}^i y_{t'}^i + s^i \quad i = [1, \dots, NI], t' = 1, 2 \quad (2)$$

$$x_1^i + x_2^i \leq \tilde{d}_1^i y_1^i + \tilde{d}_2^i y_2^i + s^i \quad i = [1, \dots, NI] \quad (3)$$

$$x_1^i + x_2^i \leq \tilde{d}_1^i + s^i \quad i = [1, \dots, NI] \quad (4)$$

$$\sum_{i=1}^{NI} (x_{t'}^i + ST_{t'}^i y_{t'}^i) \leq C_{t'} \quad t' = 1, 2 \quad (5)$$

$$x, s \geq 0, y \in \{0, 1\}^{2 \times NI} \quad (6)$$

Thus, constraints (4) represent the joint capacity constraints, while constraints (1) relate the production variables  $x$  with the setup variables  $y$ . Constraints (2) and (3) are simply the  $(\ell, S)$  inequalities of [4]. Constraints (4) enforce the fact that total production within the two periods must be inferior to the total demand, unless we carry inventory *out of* the two-period interval.

Since even two-period problems are NP-complete, it is not unlikely that we can define  $\text{conv}(X^{2PL})$  explicitly by valid inequalities. Therefore, given a fractional solution  $(\bar{x}, \bar{y}, \bar{s})$ , separation over  $\text{conv}(X^{2PL})$  requires us to determine whether  $(\bar{x}, \bar{y}, \bar{s})$  can be expressed as a convex combination of extreme points of  $\text{conv}(X^{2PL})$ . This problem is equivalent to choosing a point in  $\text{conv}(X^{2PL})$  such that the distance from  $(\bar{x}, \bar{y}, \bar{s})$  to this point is minimized, which requires enumerating, at least implicitly, the extreme points of  $\text{conv}(X^{2PL})$ . We do this by column generation. Solving the pricing problem in this scheme is equivalent to optimizing over  $\text{conv}(X^{2PL})$ , which in practice is trivial. In this way we can check if  $(\bar{x}, \bar{y}, \bar{s}) \in \text{conv}(X^{2PL})$ ; if not, we can generate a valid inequality via Farkas' Lemma that cuts off the fractional point. This *local cut* is in the *convex hull closure* of the intersection of two-period relaxations. To our knowledge, such methods have not been applied to production planning problems before, and has only rarely been applied to other MIP problems.

We are in the process of compiling extensive results for multi-level, capacitated big bucket instances of realistic size. The evidence suggests that the strengthened formulations generated by our procedure are tighter than those produced by these and other authors in previous research.

## Références

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