Approximation algorithm for directed tree cover

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1 Introduction

Let $G = (V, A)$ be a (weakly) connected directed graph with a (non negative) cost function $c : A \Rightarrow \mathbb{Q}_+$ defined on the arcs. Let $c(u, v)$ denote the cost of the arc $(u, v) \in A$. A *directed tree cover* is a weakly connected subgraph $T = (U, F)$ such that

1. for every $e \in A$, F contains an arc f intersecting e, i.e. f and e have at least an end vertex in common.
2. T is a branching.

The *minimum directed tree cover problem* (DTCP) is to find a directed tree cover of minimum cost. Several related problems to DTCP have been investigated, in particular :

– its undirected counterpart, the minimum tree cover problem (TCP) and
– the tour cover problem in which T is a tour (not necessarily simple) instead of a tree. This problem has also two versions : undirected (ToCP) and directed (DToCP).

These problems have been studied in [1], [3], [2], [4] and [5]. In [2], the author pointed out that his approach for TCP can be extended to give a 2-approximation algorithm for the unweighted case of DTCP but falls short once arbitrary costs are allowed. Hence the general DTCP remains open.

2 Results

In this paper, we introduce the *minimum r-branching cover problem* which is to find a minimum directed tree cover with a specific root r. Hence, an α-approximation algorithm for DTCP can be obtained from an α-approximation algorithm for the minimum r-branching cover problem by applying this latter $|V|$ times.
We then show that the weighted Set Cover Problem (SCP) is a special case of the minimum r-branching cover problem, consequently using the known complexity result for SCP we derive the following.

Théorème 1 *Let $D^+ = \max(|\delta^+(v)|$ where $v \in V)$, the maximum outgoing degree of the nodes in G then :*

– *If there exists a $c\ln(D^+)$-approximation algorithm for the minimum r-branching cover problem where $c < 1$ then $NP \subseteq DTIME(n^{\{O(\log^k(D^+))\}})$.*

- *There exists some* $0 < c < 1$ *such that if there exists a* $c \log(D^+)$*-approximation algorithm for the minimum* r*-branching cover problem, then* $P = NP$.

Thus in some sense, our approximation for DTCP seems to be the best possible. We present first an integer formulation for DTCP inspired from the one in [3] designed originally for the TCP. The formulation is as follows : for a fixed root r, define $\mathcal{F}$ to be the set of all subsets S of $V \setminus \{r\}$ such that S induces at least one arc of A,

$$\mathcal{F} = \{S \subseteq V \setminus \{r\} \mid A(S) \neq \emptyset\}.$$

Let T be the arc set of a directed tree cover of G containing r, T is thus a branching rooted at r. Now for every $S \in \mathcal{F}$, at least one node, saying v, in S should belong to $V(T)$. By definition of directed tree cover there is a path from r to v in T and as $r \notin S$, this path should contain at least one arc in $\delta^-(S)$. This leads to the following IP formulation for the minimum r-branching cover.

$$\min \sum_{e \in A} c(e) x_e$$

$$\sum_{e \in \delta^-(S)} x_e \geq 1 \text{ for all } S \in \mathcal{F}$$

$$x \in \{0,1\}^A.$$

Based on the linear programming relaxation of this formulation, we design an algorithm which is a composition of 2 phases :

- Phase I is of a primal-dual style which tries to cover the sets $S \in \mathcal{F}$ such that $|S| = 2$. We obtain, after Phase I, a partial solution T_1 and a dual feasible solution y_1 such that the cost of T_1 is at most 2 times the cost of y_1. Note that the partial solution T_1 is a cover but not necessary weakly connected branching.
- Phase II works with the reduced costs issued from Phase I and preserves the partial solution T_1. The problem is to make T_1 weakly connected. Phase II tranforms this problem to a kind of Set Cover Problem and solve it by a greedy algorithm. Phase II outputs a set of arcs T_2 completing with T_1 to form a r-branching cover and a dual feasible solution y_2 such that the cost of T_2 (which is computed using the reduced costs issued from Phase I) is at most $\ln(D^+)$ times the cost of y_2.

Hence, we have the following theorem

Théorème 2 *DTCP can be approximated in polynomial time within a factor of* $\max(2, \ln(D^+))$ *with* $D^+ = \max(|\delta^+(v)|$ *where* $v \in V)$.

Références

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