Stochastic Shortest Path Problem with Delay Excess Penalty

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1 Introduction

The (deterministic) Restricted Shortest Path Problem ($RSSP$, sometimes also called *Delay Constrained Least Cost Routing Problem*) is a well studied extension of the famous Shortest Path Problem. The problem consists in finding a shortest path between two vertices of a directed or indirected graph due to further constraints. Unlike the Shortest Path Problem, the general $RSSP$ is NP-hard.
We study a particular stochastic version of the $RSSP$, the Stochastic Shortest Path Problem with Delay Excess Penalty. The objective is to find a shortest (or minimum cost) path in a given network due to a delay constraint. We assume the arc delays to be random, following independent normal distributions, while the arc costs are kept deterministic. The model we propose is a simple recourse model. This means that we handle the randomness of the delay by introducing a penalty that occurs in case the delay constraint is not satisfied by the actual delay on the chosen path. The payment of the penalty can be seen as a second stage action, especially if we interprete it as the "purchase of additional delay", which explains the term *simple recourse.* This particular stochastic variant of the $RSSP$ has been studied previously (see [2]). However, the authors assumed discrete distributions for the delay, while in our case the distributions are continuous.

2 Problem Formulation

Let $G = (V, A)$ be a directed, simple graph without directed cycles and let $s, t \in V$ be fixed. We assign to every arc $a \in A$ a (deterministic) cost $c(a)$ as well as a normally distributed random variable $\delta(a)$ representing the stochastic delay. The Stochastic Shortest Path Problem with Delay Excess Penalty ($SSPD$) consists in finding a directed path from s to t such that the cost of the path plus the expected delay cost are minimal. The delay cost is based on a penalty d that has to be paid per time unit that the delay excesses a given fix delay $D \geq 0$. The corresponding mathematical formulation is as follows :

$$\text{(SSPD)} \quad \min_{x\in\{0,1\}^{|A|}} \quad \mathbb{E}[j(x,\delta)] := \sum_{a\in A} c(a)x_a + d\cdot \mathbb{E}[[\sum_{a\in A}\delta(a)x_a - D]^+] \tag{1a}$$

$$\text{s.t.} \sum_{wv\in A} x_{wv} - \sum_{vw\in A} x_{vw} = \begin{cases} -1 & \text{if} \quad v=s, \\ 1 & \text{if} \quad v=t, \\ 0 & \text{else.} \end{cases} \tag{1b}$$

where $[\cdot]^+ = max(0,\cdot)$. Constraints (1b) are combined in the linear constraint $Mx = b$.

3 Problem Solving Method

The idea to solve problem (1) is the following : To search the graph for the optimal path we apply a branch-and-bound algorithm on the search space $\mathcal{P}$ of directed paths from s to t. Here we use the fact that due to the choice of normally distributed delays, the objective function (1a) has a deterministic equivalent formulation and can thus be evaluated exactly.
In order to sort out valueless subsets of $\mathcal{P}$ we calculate lower bounds by solving the corresponding relaxed, i.e. continuous version of problem (1) (see following subsection).

3.1 Solving the relaxed SSPD

Due to the convexity of the relaxation of problem (1) (we replace the constraint $x \in \{0,1\}^n$ by $x \in [0,1]^n$), we can apply a projected stochastic gradient method (see e.g. [1]). The idea is the following : in each iteration k we first relax the linear constraints and calculate the gradient $\nabla_x j(x^{k-1},\overline{\delta})$. Here x^{k-1} is the previously computed, feasible solution vector and $\overline{\delta}$ is a realization of the random vector δ that is regenerated at each iteration. Remark that there exists a null set $\mathcal{N} \in \mathbb{R}^{|A|} \times \mathbb{R}^{|A|}$ where j is not differentiable. However, this null set can be neglected, as we are in the end interested in the *expectation* of the gradient of j.
The obtained gradient of j is projected on the null space of the matrix M. x^k is then computed as usual, i.e. as the sum of x^{k-1} and the projected gradient times the previously fixed step size. In case this update leads to negative components of x^k, we shorten the step size appropriately and it can be shown that this modification is sufficient to obtain $x_i^k \in [0,1]$ for all $i \in \{1,\dots,n\}$. x^k is therefore a new feasible solution to the relaxation of problem (1).
We further introduce a so called *active set* of additional equality constraints of the form $[x_i = 0]$ in order to stabilize the convergence of the algorithm and reduce the computing time. In contrary to most active set strategies in deterministic optimization, we allow equalities to be removed from the set following a randomized procedure.

Références

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